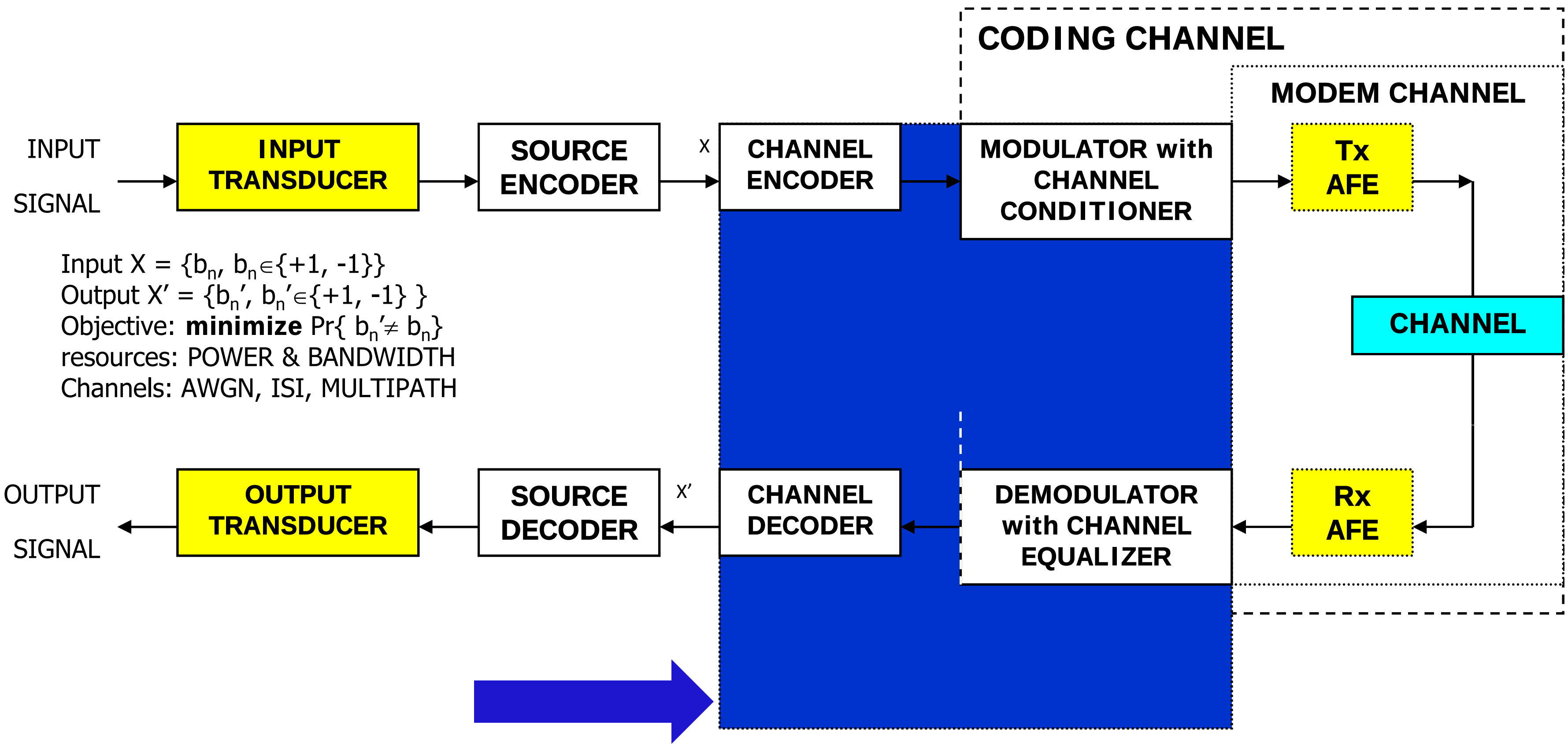
Tho Le-Ngoc, Winter 2008

Basic Digital Modulation Techniques:

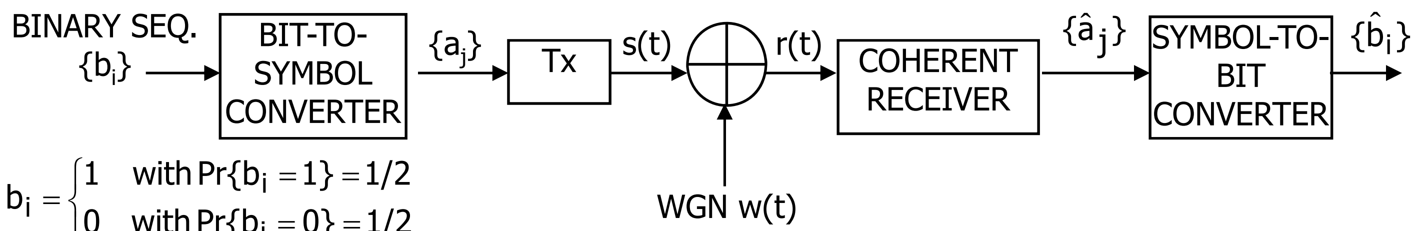
Digital Transmission in AWGN •
Optimum Receiver • Probability of Error •
Digital Modulation Techniques: ASK, PSK, QAM, FSK •

Elements of a digital communications link



Input X = $\{b_n, b_n \in \{+1, -1\}\}$
Output X' = $\{b_n', b_n' \in \{+1, -1\}\}$
Objective: **minimize** Pr$\{b_n' \neq b_n\}$
resources: POWER & BANDWIDTH
Channels: AWGN, ISI, MULTIPATH

TIME-LIMITED SIGNALLING SCHEMES IN AN AWGN ENVIRONMENT



$$b_i = \begin{cases} 1 & \text{with } \Pr\{b_i = 1\} = 1/2 \\ 0 & \text{with } \Pr\{b_i = 0\} = 1/2 \end{cases}$$

- Input binary sequence: $\{b_i\}$, transmitted in $|t-iT_b| \leq T_b/2$, T_b: bit interval
- Every m bits: grouped to form 1 symbol → $M=2^m$ possible symbols,
- Symbol sequence: $\{a_j\}$, $a_j=A_k$, $k=1,2,3,\ldots, 2^m$ with $\Pr\{a_j=A_k\}=1/M$.
- a_j is transmitted in the interval $|t-jT_s| \leq T_s/2$, $T_s=mT_b$: symbol interval.
- TRANSMITTER: generates $s(t) = g_k(t-jT_s)$ for $|t-jT_s| \leq T_s/2$ if $a_j = A_k$
  i.e., ONE-TO-ONE MAPPING
- AWGN CHANNEL: $r(t)= s(t)+w(t)$ where $w(t)$: white Gaussian noise, zero-mean, variance $N_o/2$.

VECTOR REPRESENTATION OF TIME-LIMITED SIGNALS

$s(t) = g_k(t-jT_s)$ for $|t-jT_s| \leq T_s/2$ if $a_j = A_k$: one-to-one correspondence

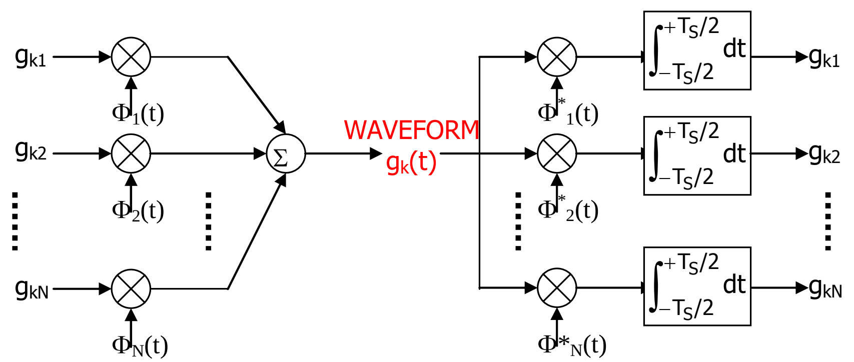
$g_k(t)$: signaling element $g_k(t)=0$ for $|t|>T_S/2$: **time-limited** and $\int_{-T_s/2}^{+T_s/2} |g_k(t)|^2 \, dt = E_k < +\infty$

We want to find N ($N \leq M$) orthonormal functions $\Phi_i(t)$, i=1,2,..., N so that $g_k(t) = \sum_{i=1}^{N} g_{ki}\Phi_i(t)$

signal element: waveform $g_k(t)$ can be represented by N-dimensional vector $\underline{g}_k = (g_{k1}, \ldots, g_{kN})$

ORTHONORMAL FUNCTIONS: $\Phi_i(t)=0$ for $|t|>T_S/2$ and $\int_{-\infty}^{+\infty} \Phi_i(t)\Phi_j^*(t)dt = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{f } i \neq j \end{cases}$

$\Phi_j^*(t)=\Phi_j(t)$ if $\Phi_i(t)$: REAL, $\{\Phi_i(t)$, i=1,2,...,N$\}$ forms an ORTHONORMAL BASIS of N dimensions.
waveform $g_k(t)$ can be represented by N-dimensional vector
$\underline{g}_k = (g_{k1}, \ldots, g_{kN})$: $g_k(t) = \sum_{i=1}^{N} g_{ki}\Phi_i(t)$ where $g_{ki} = \int_{-T_S/2}^{+T_S/2} g_k(t).\Phi_i^*(t)dt$



IN TRANSMITTER

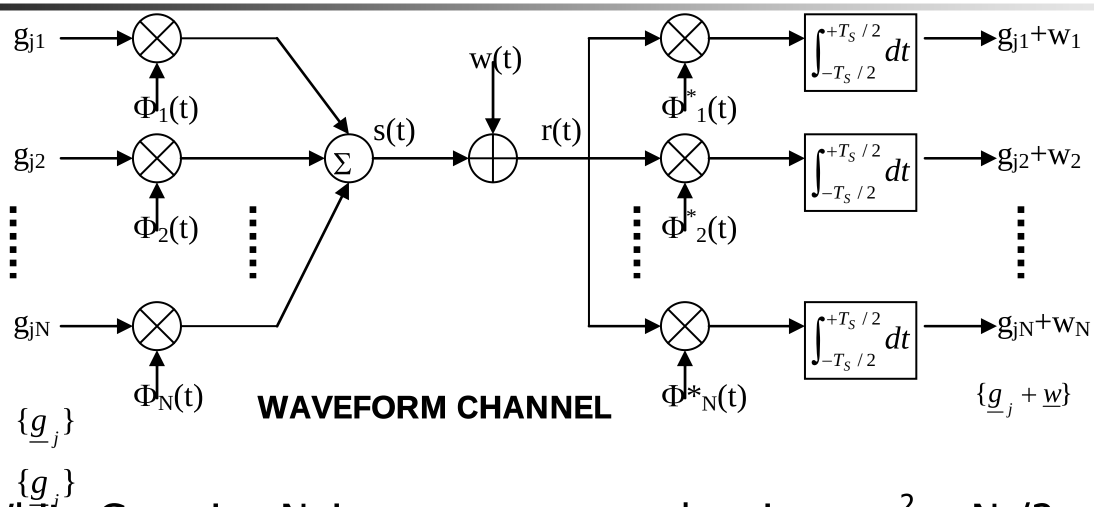
$g_k(t) = \sum_{i=1}^{N} g_{ki}\Phi_i(t)$

IN RECEIVER

$\underline{g}_k = (g_{k1}, g_{k2}, \ldots g_{kN})$,

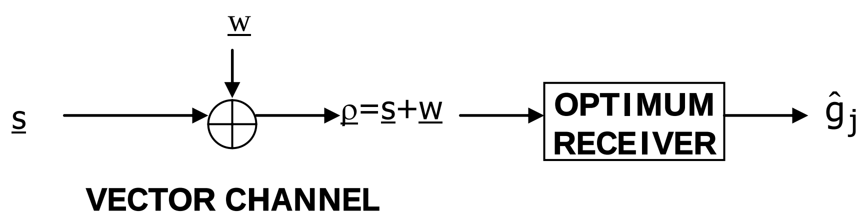
$g_{ki} = \int_{-T_S/2}^{+T_S/2} g_k(t).\Phi_i^*(t)dt$

VECTOR CHANNEL



w(t): White Gaussian Noise, zero mean and variance $\sigma^2_W = N_0/2$,

→ **noise components** w_k's, k=1,2,...,N are Gaussian with zero mean and variance of $N_0/2$.
They are also statistically independent.



$$p_{w_k}(x_k) = \frac{1}{(\pi N_0)^{1/2}} \exp\left[-\frac{x_k^2}{N_0}\right], \quad p_{\underline{w}}(\underline{x}) = \prod_{k=1}^{N} p_{w_k}(x_k) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\frac{\|\underline{x}\|^2}{N_0}\right], \quad \|\underline{x}\|^2 = \sum_{k=1}^{N} x_k^2$$

OPTIMUM RECEIVER

Minimize Pr{$\hat{a}_j$ = A_l | a_j = A_k, k≠l}= minimize Pr{$\underline{\hat{s}}$ = $\underline{s}_l$ | $\underline{s}$ = $\underline{g}_k$, k≠l}, or

maximize Pr{$\hat{a}_j$ = A_k | a_j = A_k} = maximize Pr{$\underline{\hat{s}}$ = $\underline{g}_k$ | $\underline{s}$ = $\underline{g}_k$} = maximize Pr{$\underline{\hat{s}}$ = $\underline{s}_k$ | $\underline{\rho}$= $\underline{g}_k$ +$\underline{w}$}

optimum maximum a posteriori probability (MAP) receiver:

Max Pr{$\underline{\hat{s}}$ = $\underline{s}_k$ | $\underline{\rho}$= $\underline{g}_k$ +$\underline{w}$}:

Procedure: For a receiver vector $\underline{\rho}$, calculate all Pr{ $\underline{g}_i$ | $\underline{\rho}$} i=1,2,...,M

Select the index k corresponding to the MAXIMUM value Pr{ $\underline{g}_k$ | $\underline{\rho}$} and declare $\underline{\hat{s}} = \underline{g}_k$

Using the Bayes rule: Pr{A|B}.p(B)=p(B|A).Pr{A} for A: discrete, B: continuous

p(B): probability density function (pdf) of B

$$\Rightarrow \Pr\{\underline{g}_i \mid \underline{\rho}\} = \frac{p\{\underline{\rho} \mid \underline{g}_i\}.\Pr\{\underline{g}_i\}}{p\{\underline{\rho}\}} \Rightarrow \underset{\text{wrt}\,\underline{g}_i}{\text{Max}} \Pr\{\underline{g}_i \mid \underline{\rho}\} \equiv \underset{\text{wrt}\,\underline{g}_i}{\text{Max}} \left[p\{\underline{\rho} \mid \underline{g}_i\}.\Pr\{\underline{g}_i\}\right]$$

If Pr{$\underline{g}_i$}=1/M, i=1,2,...,M $\Rightarrow \underset{\text{wrt}\,\underline{g}_i}{\text{Max}} \Pr\{\underline{g}_i \mid \underline{\rho}\} \equiv \underset{\text{wrt}\,\underline{g}_i}{\text{Max}} \left[p\{\underline{\rho} \mid \underline{g}_i\}\right]$

optimum receiver = maximum likelihood (ML) receiver

For $\underline{g}_i$, i=1,2,...,M, calculate all p{$\underline{\rho}$ | $\underline{g}_i$}. Select k corresponding to the largest p{$\underline{\rho}$ | $\underline{g}_k$}

→ ML receiver selects $\underline{g}_k$, the most likely signaling vector in producing $\underline{\rho}$

MAP and ML in AWGN channel

Transmitter: sends $\mathbf{s}_i=\mathbf{g}_k$ with a priori probability of $\Pr\{\mathbf{g}_k\}$.

Receiver: From the received sample $\mathbf{r}_i = \mathbf{s}_i + \mathbf{w}$ where $\mathbf{w}$: Gaussian (0, $N_o/2$), guess $\mathbf{s}_i$

$$\text{AWGN}(0,N_o/2)\text{CHANNEL:} if\ \mathbf{g}_k\ sent \rightarrow \boldsymbol{\rho} = \mathbf{g}_k + \mathbf{w}\ k{=}1 \text{ or } 0,\ p(\boldsymbol{\rho}\,|\,\mathbf{g}_k) = \frac{1}{[\pi N_o]^N}\exp\left(-\frac{\|\mathbf{r}_i - \mathbf{g}_k\|^2}{N_o}\right)$$

$$\rightarrow \ln\left[\,p(\boldsymbol{\rho}\,|\,\mathbf{g}_k)\right] = -\frac{\|\mathbf{r}_i - \mathbf{g}_k\|^2}{N_o} - 0.5N\ln\left[\pi N_o\right]$$

Maximum A Posteriori (MAP): Choose $\mathbf{g}_k$ corresponding to $\max \Pr\{\mathbf{g}_k\,|\,\boldsymbol{\rho}\} \equiv \max\, p(\boldsymbol{\rho}\,|\,\mathbf{g}_k)\Pr\{\mathbf{g}_k\}$

$$\rightarrow \max \ln\left[p(\boldsymbol{\rho}\,|\,\mathbf{g}_k)\Pr\{\mathbf{g}_k\}\right] = \max\left[\ln p(\boldsymbol{\rho}\,|\,\mathbf{g}_k) + \ln\Pr\{\mathbf{g}_k\}\right] = \min\left[\frac{\|\mathbf{r}_i - \mathbf{g}_k\|^2}{N_o} + \left(0.5N\ln\left[\pi N_o\right] - \ln\Pr\{\mathbf{g}_k\}\right)\right]$$

Maximum Likelihood (ML): Choose $\ln\Pr\{\mathbf{g}_k\}$corresponding to $\max\, p(\boldsymbol{\rho}\,|\,\mathbf{g}_k) = \max\ln\left[\,p(\boldsymbol{\rho}\,|\,\mathbf{g}_k)\right] \equiv \min\|\mathbf{r}_i - \mathbf{g}_k\|^2$

i.e., $\rightarrow$ For general M, Select k corresponding to the minimum Euclidean distance $\|\mathbf{r}_i - \mathbf{g}_k\|^2$ among all $\|\mathbf{r}_i - \mathbf{g}_m\|^2$

$\rightarrow$ when $\Pr\{\mathbf{g}_k\} = 1/M \Rightarrow$ (MAP): $\max\ln\Pr\{\mathbf{g}_k\,|\,\boldsymbol{\rho}\} \equiv \min\|\mathbf{r}_i - \mathbf{g}_k\|^2$ (ML)

MAP and ML: example of binary case (M=2)

Transmitter: From binary sequence $\{b_i\}$, b_i =1 or 0 with a priori probability $\Pr\{b_i=1\}$ and $\Pr\{b_i=0\}$, respectively, sends $s_i=g_1$ if b_i =1 or $s_i=g_0$ if b_i =0.

Receiver: From the received sample $r_i=s_i+w$ where w: Gaussian (0, $N_o/2$), guess b_i

Maximum A Posteriori (MAP): Choose $\hat{b}_i=1$ if $\Pr\{b_i=1|r_i\}>\Pr\{b_i=0|r_i\}$, otherwise choose $\hat{b}_i=0$

$$\Lambda_{MAP}(r_i)\equiv\ln\left[\frac{\Pr\{b_i=1|r_i\}}{\Pr\{b_i=0|r_i\}}\right]\rightarrow\hat{b}_i=\begin{cases}1, \text{ if } \Lambda_{MAP}(r_i)>0\\0, \text{ if } \Lambda_{MAP}(r_i)<0\end{cases}$$

From Bayes rule, $\Lambda_{MAP}(r_i)\equiv\ln\left[\frac{p(r_i|b_i=1)\Pr\{b_i=1\}/p(r_i)}{p(r_i|b_i=0)\Pr\{b_i=0\}/p(r_i)}\right]=\ln\left[\frac{p(r_i|b_i=1)}{p(r_i|b_i=0)}\right]+\ln\left[\frac{\Pr\{b_i=1\}}{\Pr\{b_i=0\}}\right]$

$$\rightarrow\Lambda_{MAP}(r_i)=\Lambda_{ML}(r_i)+\Lambda_P(b_i),\quad \Lambda_{ML}(r_i)\equiv\ln\left[\frac{p(r_i|b_i=1)}{p(r_i|b_i=0)}\right],\quad \Lambda_P(b_i)\equiv\ln\left[\frac{\Pr\{b_i=1\}}{\Pr\{b_i=0\}}\right]$$

Maximum Likelihood (ML): Choose $\hat{b}_i=1$ if $p(r_i|b_i=1)>p(r_i|b_i=0)$, otherwise choose $\hat{b}_i=0$

$$\rightarrow\hat{b}_i=\begin{cases}1, \text{ if } \Lambda_{ML}(r_i)>0\\0, \text{ if } \Lambda_{ML}(r_i)<0\end{cases}$$

when $\Pr\{b_i=1\}=\Pr\{b_i=0\}\Rightarrow\Lambda_P(b_i)=0\Rightarrow\Lambda_{MAP}(r_i)=\Lambda_{ML}(r_i)$

MAP and ML: example of binary case (M=2) in AWGN

AWGN(0,$N_o/2$): k=1 or 0, $p(r_i \mid b_i = k) = \frac{1}{\sqrt{\pi N_o}} \exp\left(-\frac{(r_i - g_k)^2}{N_o}\right)$

$$\rightarrow \ln\left[p(r_i \mid b_i = k)\right] = -\frac{(r_i - g_k)^2}{N_o} - 0.5\ln\left[\pi N_o\right]$$

$$\rightarrow \Lambda_{ML}(r_i) \equiv \ln\left[\frac{p(r_i \mid b_i = 1)}{p(r_i \mid b_i = 0)}\right] = \frac{(r_i - g_0)^2 - (r_i - g_1)^2}{N_o},$$

$$\Lambda_{MAP}(r_i) \equiv \ln\left[\frac{\Pr\{b_i = 1 \mid r_i\}}{\Pr\{b_i = 0 \mid r_i\}}\right] = \frac{(r_i - g_0)^2 - (r_i - g_1)^2}{N_o} + \Lambda_P(b_i),\ \Lambda_P(b_i) \equiv \ln\left[\frac{\Pr\{b_i = 1\}}{\Pr\{b_i = 0\}}\right]$$

Maximum Likelihood (ML): Choose $\hat{b}_i$=1 if $p(r_i \mid b_i = 1) > p(r_i \mid b_i = 0)$, otherwise choose $\hat{b}_i = 0$

$$\rightarrow \hat{b}_i = \begin{cases} 1, \text{ if } (r_i - g_0)^2 > (r_i - g_1)^2 \\ 0, \text{ if } (r_i - g_0)^2 < (r_i - g_1)^2 \end{cases}$$

For general M, Select k corresponding to the minimum Euclidean distance $\|\mathbf{r}_i - \mathbf{g}_k\|^2$ among all $\|\mathbf{r}_i - \mathbf{g}_m\|^2$

PROBABILITY OF ERROR

For $\underline{g}_k$ sent, the ML receiver makes an error if it decides $\hat{\underline{s}} = \underline{g}_l, l \neq k$. This event occurs if and only if $\|\underline{\rho} - \underline{g}_k\|^2$ is not minimum i.e. $\|\underline{\rho} - \underline{g}_k\|^2 > \|\underline{\rho} - \underline{g}_n\|^2$ for some $n \neq k$

In the N-dimensional observation space Z, the optimum receiver establishes M disjoint zones Z_i as follows

$$\bigcup_{i=1}^{M} Z_i = Z \quad Z_i \bigcap Z_j = \emptyset \quad \text{for } i \neq j \qquad Z_i = \{\underline{\rho} : \|\underline{\rho} - \underline{g}_i\|^2 \text{ is minimum}\}$$

For an observation vector $\underline{\rho}$ if $\underline{\rho} \in Z_k$ then the ML receiver declares that $\hat{a} = A_k$ was sent.

Therefore the average probability of error is $P_e = \sum_{k=1}^{M} \Pr\{\underline{\rho} \notin Z_k \mid A_k\} \Pr\{A_k\}$

$$\text{for } \Pr\{A_k\} = \frac{1}{M}, P_e = \frac{1}{M}\sum_{k=1}^{M} \Pr\{\underline{\rho} \notin Z_k \mid A_k\} = \frac{1}{M}\sum_{k=1}^{M}[1 - \Pr\{\underline{\rho} \in Z_k \mid A_k\}] = 1 - \underbrace{\frac{1}{M}\sum_{k=1}^{M} \Pr\{\underline{\rho} \in Z_k \mid A_k\}}_{P_c = \Pr\{\text{correct decision}\}}$$

$$\text{where } \Pr\{\underline{\rho} \in Z_k \mid A_k\} = \int_{\substack{\|\underline{\rho}-\underline{g}_k\|^2 \\ \text{is minimum}}} p_w(\underline{\rho} - \underline{g}_k) d\underline{\rho} = \frac{1}{(\pi N_0)^{N/2}} \int_{\substack{\|\underline{\rho}-\underline{g}_k\|^2 \\ \text{is minimum}}} \exp\left(-\frac{\|\underline{\rho} - \underline{g}_k\|^2}{N_0}\right) d\underline{\rho}$$

VECTOR REPRESENTATION FOR A GENERAL BINARY TIME-LIMITED SIGNALING SCHEME

In the interval $|t-nT_b| \le T_b/2$,

$$s(t) = \begin{cases} g_1(t-nT_b) & \text{if } a_n = 1, \text{with a priori probability of } 1/2 \\ g_2(t-nT_b) & \text{if } a_n = 0, \text{with a priori probability of } 1/2 \end{cases}$$

For a general binary signaling scheme with time-limited, finite-energy elements, $g_1(t)$ and $g_2(t)$, for a simple 1-D receiver design, we can select the orthonormal basis with

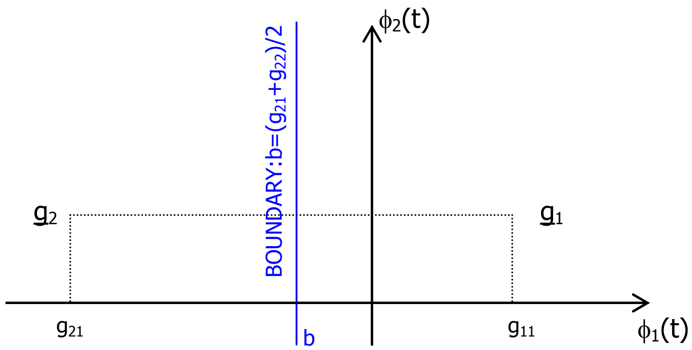
$$\Phi_1(t) = \frac{g_1(t)-g_2(t)}{\sqrt{E_\Delta}}, E_\Delta = \int_0^{T_b} |g_1(t)-g_2(t)|^2 dt = \int_0^{T_b} |g_1(t)|^2 dt + \int_0^{T_b} |g_2(t)|^2 dt - 2\int_0^{T_b} g_1(t).g_2(t)dt$$

$$E_\Delta = E_1 + E_2 - 2E_{12}, \quad E_{12} = \int_0^{T_b} g_1(t).g_2(t)dt \quad \textit{for} \quad g_1(t), g_2(t): \text{real-valued}$$

$$d^2 = E_1 + E_2 - 2E_{12} = 2E_b(1-\gamma_{12}) \ge 0 \text{ where } E_b = \frac{1}{2}(E_1+E_2): \text{average energy per bit}$$

$$\gamma_{12} = \frac{E_{12}}{E_b}: \text{correlation coefficient between } \underline{g}_1, \underline{g}_2 \ (g_1(t) \,\&\, g_2(t)) \quad \textit{and} \quad -1 \le \gamma_{12} \le 1$$

Then, $\Phi_2(t) = \dfrac{(E_2-E_{12})g_1(t) + (E_1-E_{12})g_2(t)}{\sqrt{E_a}}, E_a = \int_0^{T_b} |(E_2-E_{12})g_1(t) + (E_1-E_{12})g_2(t)|^2 dt$



$g_1(t) = g_{11}\phi_1(t) + g_{12}\phi_2(t)$ and $g_2(t) = g_{21}\phi_1(t) + g_{22}\phi_2(t)$,

$g_{11} = (E_1 - E_{12})/d$, $g_{21} = -(E_2 - E_{12})/d$

$g_{12} = g_{22} = (E_1E_2 - E_{12}^2)/E_a^{1/2}$

$d^2 = E_\Delta = E_1 + E_2 - 2E_{12} = ||\underline{g}_1 - \underline{g}_2||^2$

PROBABILITY OF ERROR OF BINARY TRANSMISSION IN AN AWGN ENVIRONMENT

For antipodal signaling: $g_1(t)=-g_2(t)$, $-E_{12}=E_1=E_2$, $d^2=4E_1$
$\phi_1(t)=g_1(t)/E_1^{1/2}$ $\phi_2(t)=0$: one dimension, $g_{11}=-g_{21}=E_1^{1/2}$
For orthogonal signaling: $\int_0^{T_b} g_1(t).g_2^*(t)dt=0$ $E_{12}=0$, $d^2=E_1+E_2$
$g_{11}=E_1/[E_1+E_2]^{1/2}$ $g_{21}=-E_2/[E_1+E_2]^{1/2}$ $g_{12}=g_{22}=[E_1E_2/(E_1+E_2)]^{1/2}$

For $\underline{g}_k$ transmitted (k=1 or 2), receive $\underline{r}=\underline{g}_k+\underline{n}$, where AWGN $\underline{n}=(w_1,w_2)$
w_1,w_2: independent Gaussian with zero mean and variance: $N_o/2$.
For $\underline{g}_1$ transmitted, error if $w_1<-d/2$. For $\underline{g}_2$ transmitted, error if $w_1>d/2$. w_2 and hence r_2 are **irrelevant**. The Rx considers only r_1 in detection.

$$\Rightarrow P_e=\Pr\{error\,|\,\underline{g}_1\ sent\}\Pr\{\underline{g}_1\ sent\}+\Pr\{error\,|\,\underline{g}_2\ sent\}\Pr\{\underline{g}_2\ sent\}$$

$$\boxed{P_e=\frac{1}{2}\mathrm{erfc}\left[\frac{d}{2\sqrt{N_0}}\right]=\frac{1}{2}\mathrm{erfc}\left[\frac{1}{2}\sqrt{\frac{d^2}{N_0}}\right]}\Rightarrow\boxed{P_e=\frac{1}{2}erfc\left[\sqrt{\frac{E_b}{2N_0}(1-\gamma_{12})}\right]}$$

$$\Pr\{error\,|\,\underline{g}_1 sent\}=\Pr\{w_1\le -d/2\}=\int_{-\infty}^{-d/2}p_w(x)dx=\int_{-\infty}^{-d/2}\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{x^2}{N_0}\right)dx=\int_{d/2}^{\infty}\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{x^2}{N_0}\right)dx$$

$$=\frac{1}{\sqrt{\pi}}\int_{\frac{d}{2\sqrt{N_0}}}^{+\infty}\exp(-v^2)dv=\frac{1}{2}erfc\left[\frac{d}{2\sqrt{N_0}}\right]\text{for } v=\frac{x}{\sqrt{N_0}},dv=\frac{1}{\sqrt{N_0}}dx, x\to d/2\Rightarrow v\to\frac{1}{2\sqrt{N_0}}, x\to+\infty\Rightarrow v\to+\infty$$

ERROR FUNCTIONS

X is called normalized zero-meanGaussian (Random) variable X: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

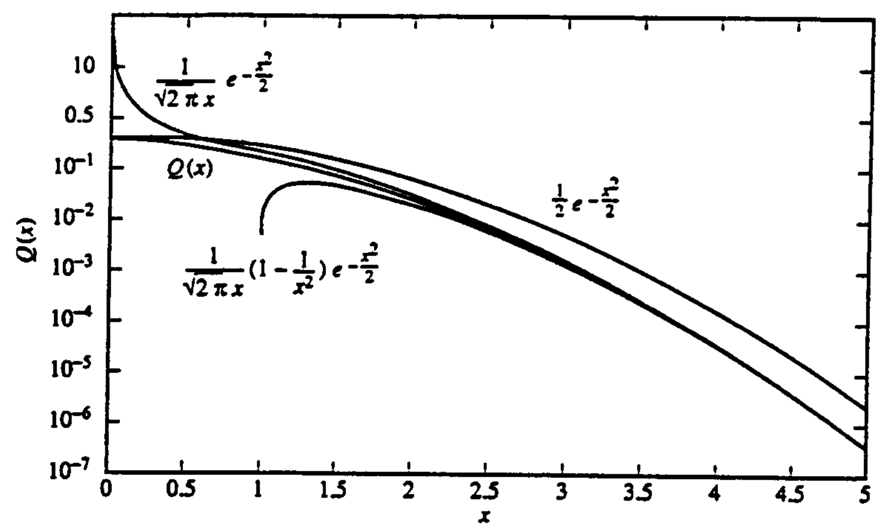
$\rightarrow$ Q-function:$Q(y) \equiv \Pr\{X > y\} = \frac{1}{\sqrt{2\pi}} \int_y^{+\infty} e^{-x^2/2} dx,$

complimentary error function: $erfc(u) \equiv \frac{2}{\sqrt{\pi}} \int_u^{+\infty} e^{-v^2} dv = \frac{2}{\sqrt{\pi}} \int_{u\sqrt{2}}^{+\infty} e^{-x^2/2} dx = 2Q(u\sqrt{2}),\ v = x/\sqrt{2}$

lower bounds: $x \geq 0,\ \frac{1}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2}\right) e^{-x^2/2} < Q(x)$

upper bounds: $x \geq 0, Q(x) < \frac{e^{-x^2/2}}{2},$

or tighter: $Q(x) < \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}$



Bounds on Q-function.

SELECTION OF $g_1(t)$, $g_2(t)$

erfc is monotone-decreasing, erfc(0)=1 → to *reduce* P_e we have to maximize $\frac{E_b}{2N_0}(1-\gamma_{12})$

A. For the same "average energy per bit to noise power density E_b/N_0, we can minimize P_e by minimize γ_{12}

$$\gamma_{12} \geq -1, \min \gamma_{12} = -1 \text{ when } g_1(t) = -g_2(t) : \text{ANTIPODAL SIGNALLING}$$

$$P_e = \frac{1}{2} erfc\sqrt{\frac{E_b}{N_0}} : \text{the BEST}$$

Example: NRZ antipodal $\Rightarrow$

$$b_i = 1 \text{ send } g_1(t) = \begin{cases} A & for\ |t - iT_b| \leq T_b/2 \\ 0 & elsewhere \end{cases}$$

$$b_i = 0 \text{ send } g_2(t) = -g_1(t)$$

B. WORST CASE: γ_{12}=1 when $g_1(t)= g_2(t)$, i.e. send the same signal for both cases. P_e=1/2, 50% correct, 50% wrong

C. Orthogonal signal: $\gamma_{12} = 0, \quad \int_{-T_b/2}^{T_b/2} g_1(t) g_2(t) dt = 0$

$$P_{e,orthogonal} = \frac{1}{2} erfc\sqrt{\frac{E_b}{2N_0}} \text{ worse than antipodal signalling}$$

M-ARY SIGNALLING SCHEME:
Union Bound on the Probability of Error

$\underline{g}_i, i = 1, 2, ..., M$ with $\Pr\{\underline{g}_i \text{ sent}\} = 1/\text{M} \rightarrow P_e = \frac{1}{M}\sum_{i=1}^{M} \Pr\{error \mid \underline{g}_i \text{ sent}\}$

For the optimum receiver,

$$\Pr\{error \mid \underline{g}_i\ sent\} = \Pr\left\{\left\|\underline{\rho} - \underline{g}_k\right\|^2 < \left\|\underline{\rho} - \underline{g}_i\right\|^2 \mid \underline{g}_i\ sent\right\} = \Pr\{\underline{\rho} \text{ is closer to at least } \underline{g}_k \text{ than to } \underline{g}_i \mid \underline{g}_i\ sent\}$$

ε_{ik} denotes the event that $\underline{\rho}$ is closer to $\underline{g}_k$ than to $\underline{g}_i$: $\Pr\{error \mid \underline{g}_i\ \ sent\} = \Pr\left\{\bigcup_{k=1,k\neq i}^{M} \varepsilon_{ik}\right\} \leq \sum_{k=1,k\neq i}^{M} \Pr\{\varepsilon_{ik}\}$

(Note that: $P\{A \cup B \cup C\} \leq \Pr\{A\} + \Pr\{B\} + \Pr\{C\}$)

Consider any pair of $\underline{g}_i, \underline{g}_k$ as the binary transmission case previously analyzed. We have obtained

$$\Pr\{\varepsilon_{ik}\} = \frac{1}{2} erfc\left[\frac{d_{ik}}{2\sqrt{N_0}}\right], \quad d_{ik}^2 = \left\|\underline{g}_i - \underline{g}_k\right\| \quad d_{ik} : \text{Euclidean distance between } \underline{g}_i, \underline{g}_k$$

$$\rightarrow \Pr\{error \mid \underline{g}_i \text{ sent}\} \leq \frac{1}{2}\sum_{k=1,k\neq i}^{M} erfc\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \rightarrow P_e \leq \frac{1}{2M}\sum_{i=1}^{M}\sum_{k=1,k\neq i}^{M} erfc\left(\frac{d_{ik}}{2\sqrt{N_0}}\right)$$

The above inequality shows that P_e is dominated by the term having the smallest distance $d_{ik,min}$. In designing the set of M signaling elements, to minimize P_e we should aim for the largest minimum Euclidean distance.

$$d_{\min} = \min_{\substack{i,k \\ i\neq k}} d_{ik} \rightarrow erfc\left[\frac{d_{ik}}{2\sqrt{N_0}}\right] \leq erfc\left[\frac{d_{\min}}{2\sqrt{N_0}}\right], \ \forall i \neq k \rightarrow P_e \leq \frac{1}{2}(M-1) erfc\left[\frac{d_{\min}}{2\sqrt{N_0}}\right]$$

Digital Modulation: General

- **Modulation:** Process by which some characteristic of a **carrier,** $c(t)$, is varied in accordance with a **modulating** wave. (IEEE Standard Dic. of E&E terms)
- In the k[th] *symbol* interval, $|t-kT_S|\le T_S/2$, send one of M possible time-limited signaling elements $g_i(t-kT_S)$ with a priori probability of $1/M$
- 1 symbol = m bits , $g_i(t)$: **modulated** signal, $i=1,2,\ldots,M=2^m$, with finite energy E_i
- Average energy per symbol: E_S, Average energy per bit: E_b , $E_S = mE_b$

$$c(t) = \sqrt{\frac{2}{T_S}}\cos\omega_c t, \quad E_c = \int_{-T_S/2}^{+T_S/2} |c(t)|^2\, dt = 1 \quad E_i = \int_{-T_S/2}^{+T_S/2} |g_i(t)|^2\, dt \quad E_S = \frac{1}{M}\sum_{i=1}^{M} E_i$$

- ASK (Amplitude Shift Keying), PSK (Phase SK), FSK (Frequency SK), APK, QAM
- **Objectives in designing a modulation scheme:**
- Bandwidth efficiency: max data rate (f_b) in a minimum channel bandwidth (BW)
- Power efficiency: minimum prob. of error for minimum transmitted power (or in terms of E_S/N_0, E_b/N_0), maximum resistance to interfering signals.
- Easy implementation: min circuit complexity
- Some of these goals pose conflicting requirements:

  →compromising the design for a certain application.

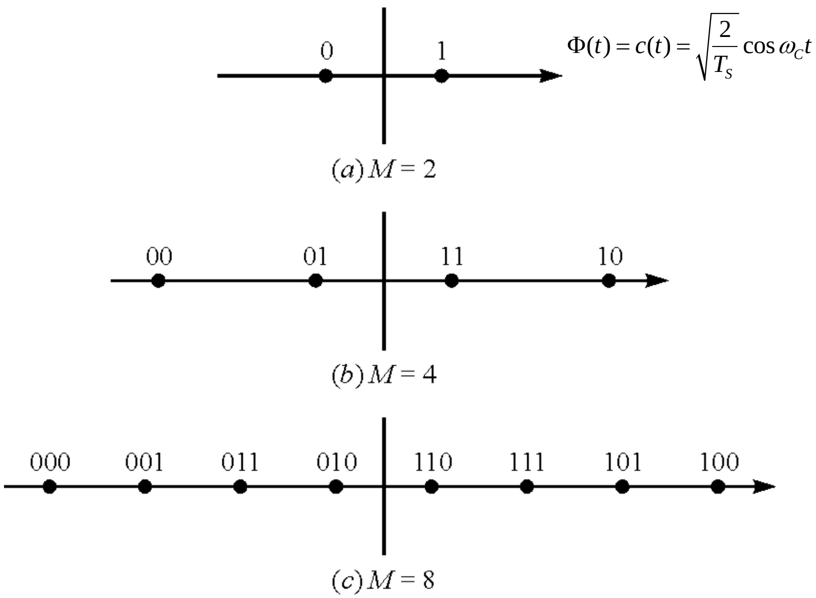
ASK (Amplitude Modulation)

$$g_i(t) = a_i\Phi(t), a_i = \left[2i-(M+1)\right]d/2, \Phi(t) = \begin{cases}\sqrt{\dfrac{2}{T_S}}\cos\omega_c t & |t|\le T_S/2 \\ 0 & elsewhere\end{cases}, \; Energy \;\; E_i = a_i^2 \;\text{ for } \omega_c >> 1 \;\; or \;\; \omega_c T_S = k\pi$$

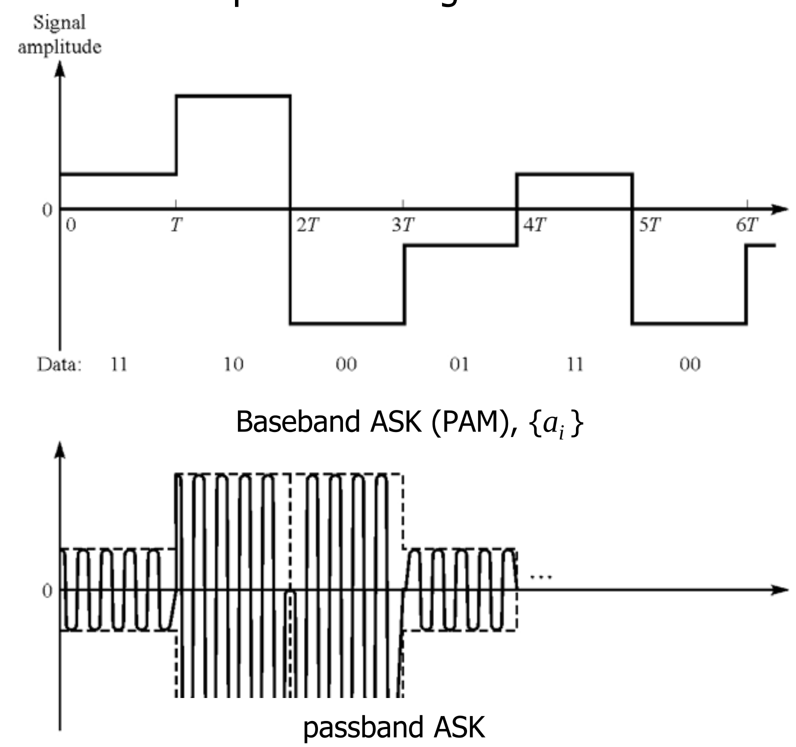
$$\Rightarrow E_S = \frac{1}{M}\sum_{i=1}^{M} a_i^2 = \frac{2}{M}\sum_{i=1}^{M/2}\left[(2i-1)\frac{d}{2}\right]^2 = \frac{d^2}{2M}\sum_{i=1}^{M/2}(2i-1)^2 = \frac{d^2(M^2-1)}{12}$$

when ω_c=0, we have baseband PAM

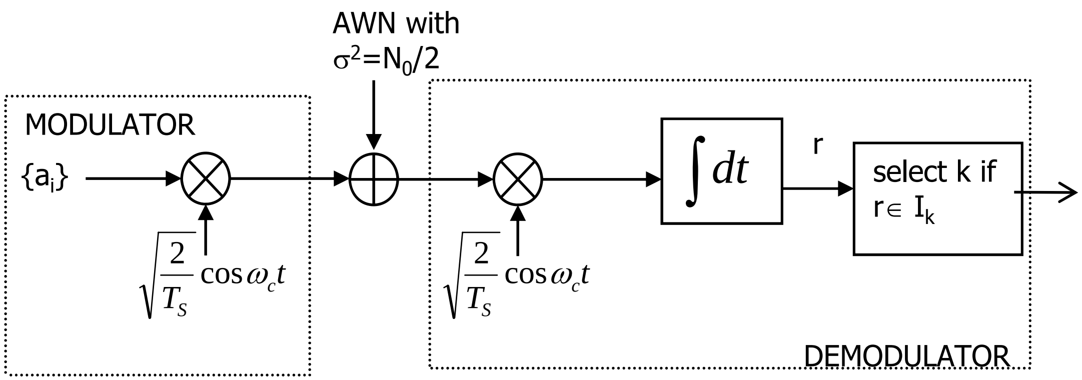
Signal constellation for M=2,4,8



Example of ASK signal with M=4



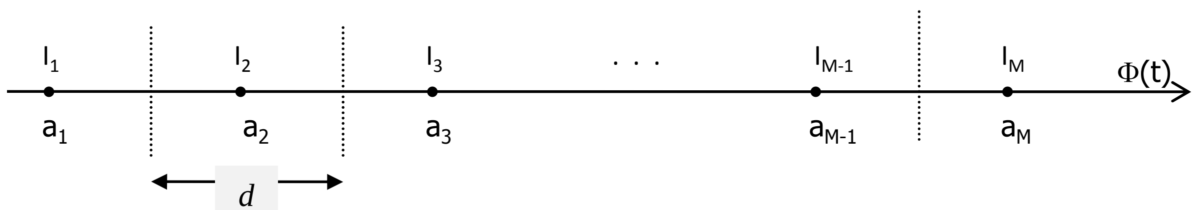
ASK: Performance



$\rho = a_i + n$,
n: Gaussian noise with $\sigma^2 = N_0/2$
if a_i sent, correct decision if $\rho \in I_i$
→correct decision,

if $-d/2 \le n \le d/2$, for $i \ne 1, M$
if $n \le d/2$, for $i=1$
if $n \ge -d/2$, for $i = M$

Pr{correct}=
(1/M)[Pr{n≤d/2}+Pr{n≥-d/2}
+(M-2)Pr{−d/2≤n≤d/2}]



Union bound: $P_e \le \frac{1}{2}(M-1)erfc\left[\frac{d}{2\sqrt{N_0}}\right] = (1/2)(M-1)erfc\sqrt{\frac{3}{M^2-1}\frac{E_S}{N_0}},\ d = \sqrt{\frac{12}{(M^2-1)}E_S}$

Exact Analysis: $P_e = \left[1 - \frac{1}{M}\right] erfc\left[\sqrt{\frac{3}{M^2}\frac{E_s}{N_0}}\right] \approx erfc\left[\sqrt{\frac{3}{M^2-1}\frac{E_s}{N_0}}\right]$ for M >> 1

ASK: Performance Analysis

$$\Pr\{n \le d/2\} = 1 - \Pr\{n \ge d/2\} = 1 - \frac{1}{2} erfc\left[\frac{d}{2\sqrt{N_0}}\right]$$

$$\Pr\{n \ge -d/2\} = 1 - \Pr\{n \le -d/2\} = 1 - \frac{1}{2} erfc\left[\frac{d}{2\sqrt{N_0}}\right] = 1 - p$$

$$\Pr\{-d/2 \le n \le d/2\} = 1 - \left[\Pr\{n \le -d/2\} + \Pr\{n \ge d/2\}\right] = 1 - 2p \text{ where } p = \frac{1}{2} erfc\left[\frac{d}{2\sqrt{N_0}}\right]$$

$$\Pr\{correct\} = \frac{1}{M}\left[2(1-p) + (M-2)(1-2p)\right] = 1 - \frac{M-1}{M}2p$$

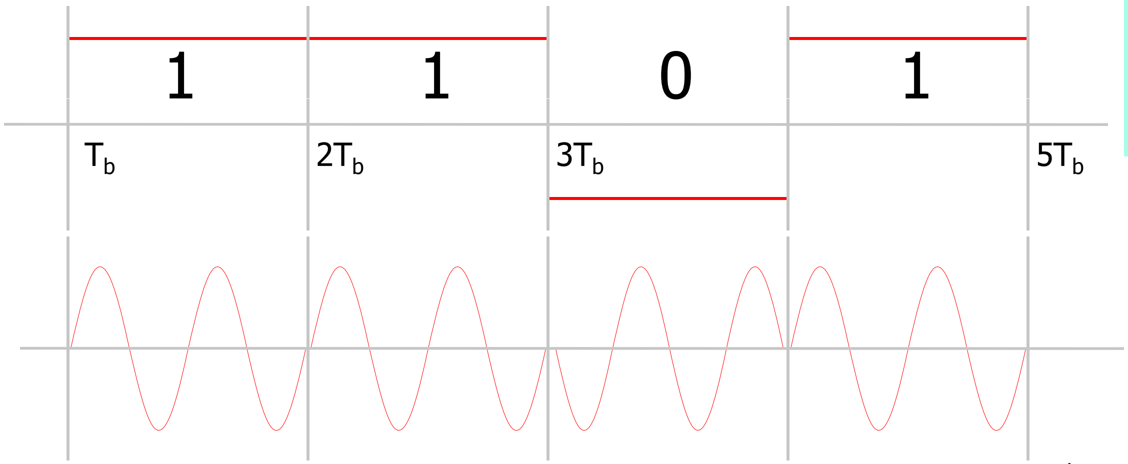
$$P_e = 1 - \Pr\{correct\} = \frac{(M-1)2p}{M} = \frac{M-1}{M} erfc\left[\frac{d}{2\sqrt{N_0}}\right]$$

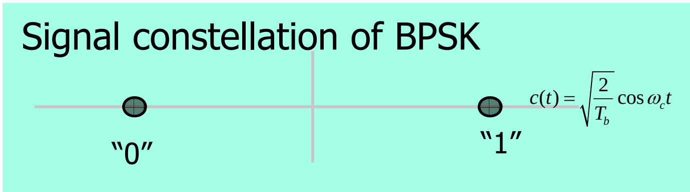
$$P_e = \left[1 - \frac{1}{M}\right] erfc\left[\sqrt{\frac{3}{M^2}\frac{E_s}{N_0}}\right] \approx erfc\left[\sqrt{\frac{3}{M^2-1}\frac{E_s}{N_0}}\right] \text{ for } M >> 1$$

Binary Phase Shift Keying (BPSK)

- Can be viewed as BPSK or APSK: Each bit is encoded in the phase of the carrier with frequency f_c: 0º for a "1" and 180º for a "0"
- In the k^{th} *symbol* interval, $|t-kT_b|\le T_b/2$, send one of *2* possible time-limited signaling elements $g_i(t- kT_b)$ with a priori probability of 1/*2*

$$c(t)=\sqrt{\frac{2}{T_b}}\cos\omega_c t \rightarrow g_1(t)=\sqrt{\frac{2E_b}{T_b}}\cos\omega_c t,\ g_0(t)=-\sqrt{\frac{2E_b}{T_b}}\cos\omega_c t=\sqrt{\frac{2E_b}{T_b}}\cos\left(\omega_c t+\pi\right)$$



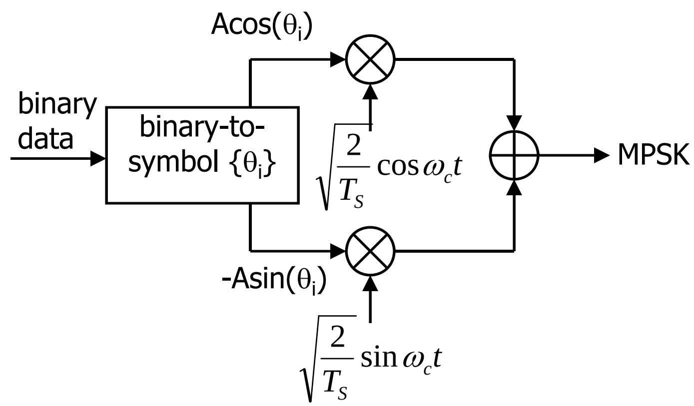


$$\Pr\{\text{bit error}\}:\quad P_b=\frac{1}{2}erfc\left[\frac{d}{2\sqrt{N_0}}\right]=\frac{1}{2}erfc\sqrt{\frac{E_b}{N_0}},\ d=2\sqrt{E_b}$$

Phase Shift Keying (PSK)

General MPSK:

$$g_i(t) = \begin{cases} \sqrt{\frac{2E_s}{T_S}}\cos[2\pi f_c t + \theta_i] = I_i\varphi_1(t) + Q_i\varphi_2(t), & |t| \le T_S/2 \\ 0 & elsewhere \end{cases}$$

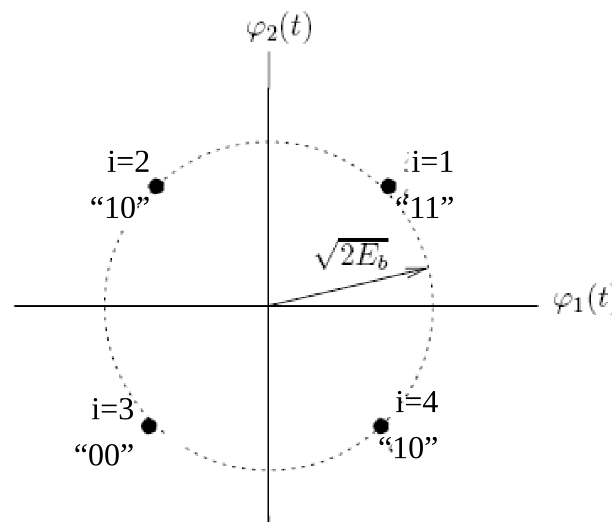


orthonormal basis function $\varphi_1(t) = \sqrt{\frac{2}{T_S}}\cos 2\pi f_c t$ & $\varphi_2(t) = \sqrt{\frac{2}{T_S}}\sin 2\pi f_c t$

$$I_i = A\cos\theta_i,\ Q_i = -A\sin\theta_i,\ A = \sqrt{E_s},\ \theta_i = (2i-1)\frac{\pi}{M},\ i = 1, 2, \ldots, M$$

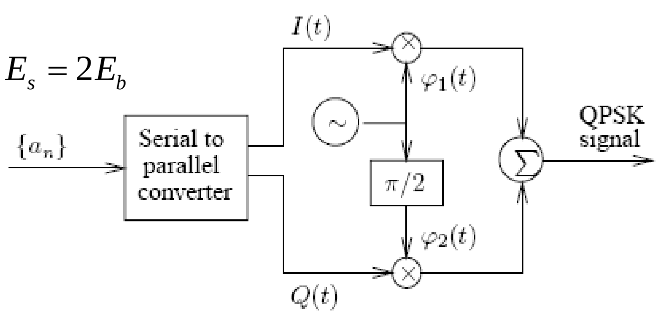
QPSK: $M=4$, can be viewed as a linear combination of an in-phase and

i	$= 1$	2	3	4
I_i	$= +\sqrt{E_s/2} = \text{"1"}$	$-\sqrt{E_s/2} = \text{"0"}$	$-\sqrt{E_s/2} = \text{"0"}$	$+\sqrt{E_s/2} = \text{"1"}$
Q_i	$= +\sqrt{E_s/2} = \text{"1"}$	$+\sqrt{E_s/2} = \text{"1"}$	$-\sqrt{E_s/2} = \text{"0"}$	$-\sqrt{E_s/2} = \text{"0"}$



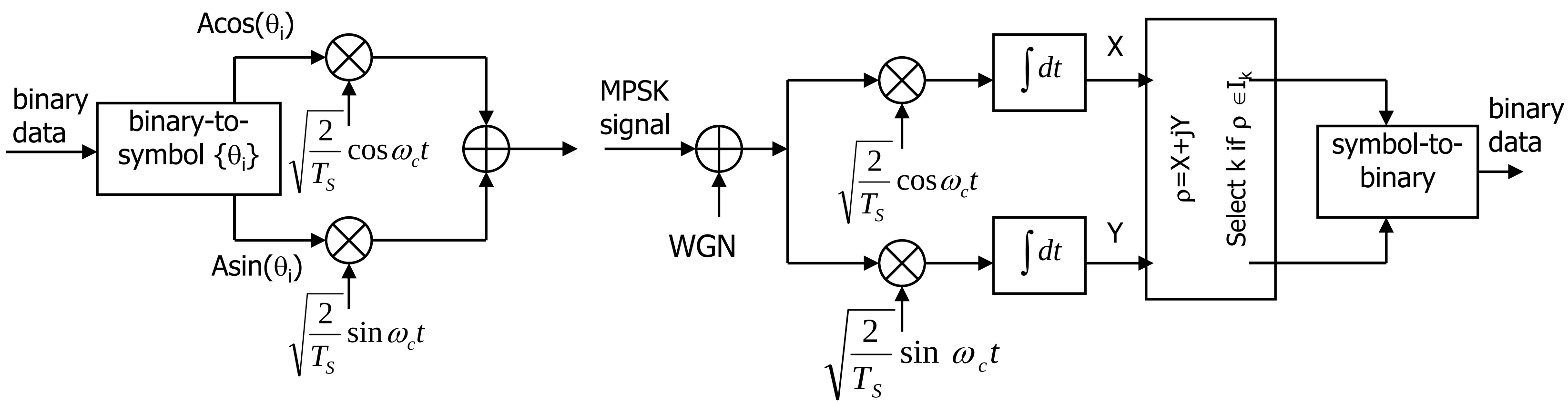
Signal constellation of QPSK

$$E_s = 2E_b$$



can be viewed as a sum of 2 BASK (or BPSK) modulated signals with in-phase and quadrature carriers.

MPSK Performance

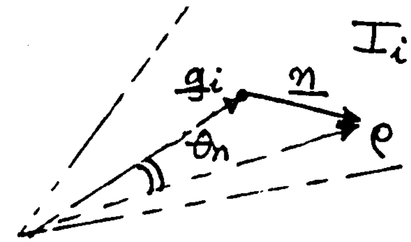


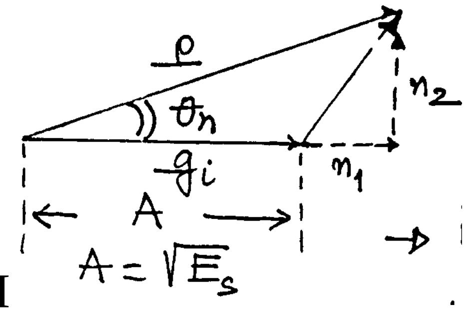
Average energy/symbol: $E_S = A^2 \quad \Rightarrow d_{\min} = 2A\sin \pi/M = \sqrt{E_S}.\sin \pi/M$

Union Bound for M-ary PSK: $P_e \le \frac{M-1}{2} erfc\left[\sin\frac{\pi}{M}\sqrt{\frac{E_S}{N_0}}\right]$

$$\text{BPSK}, M=2, P_b = P_e = \frac{1}{2} erfc\left[\sqrt{\frac{E_b}{N_0}}\right], E_S = E_b$$

MPSK Performance: Exact analysis





$$\underline{\rho} = \underline{g}_i + \underline{n},\ \underline{n} = (n_1, n_2)\text{: iid Gaussian } (0, \mathrm{N_o}/2),\ \theta_n = \tan^{-1}\left(\frac{n_2}{n_1 + \sqrt{E_S}}\right)$$

$$\rightarrow \Pr\{\text{correct}|\underline{g}_i\} = \Pr\{\underline{\rho} \in I_i\} = \Pr\left\{-\frac{\pi}{\mathrm{M}} \le \theta_\mathrm{n} \le \frac{\pi}{\mathrm{M}}\right\}, \text{for all i} = 1,2,\ldots,\mathrm{M}$$

$$p_\theta(x) = \exp\left[-\frac{E_S}{N_0}\right] + \sqrt{\frac{E_S}{\pi N_0}}\cos x \exp\left(-\frac{E_S}{N_0}\sin^2 x\right)\left[1 - \frac{1}{2}erfc\left(\sqrt{\frac{E_S}{N_0}}\cos x\right)\right]$$

$$\text{for high } \mathrm{E_S}/N_0 \text{ and } |\mathrm{x}| < \pi/2,\ erfc\left(\frac{E_S}{N_0}\cos x\right) \approx \sqrt{\frac{\mathrm{N_0}}{\pi \mathrm{E_S}}}\frac{1}{\cos x}\exp\left(-\frac{E_S}{N_0}\cos^2 x\right)$$

$$p_\theta(x) = \sqrt{\frac{E_S}{\pi N_0}}\cos x . \exp\left(-\frac{E_S}{N_0}\sin^2 x\right)$$

$$\rightarrow \Pr\{correct\} = \int_{-\pi/M}^{\pi/M} p_\theta(x)dx, \rightarrow P_e = 1 - \Pr\{correct\} \approx erfc\left[\left(\sin\frac{\pi}{M}\right)\sqrt{\frac{E_s}{N_o}}\right]$$

Square Quadrature Amplitude Modulation QAM

One case of AP(S)K: Quadrature ASK

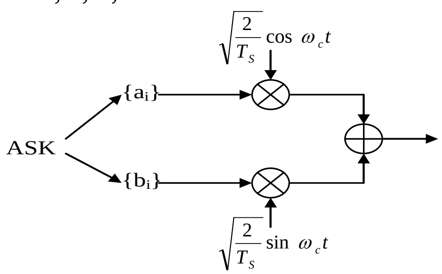
$$g_i(t) = \begin{cases} a_i\sqrt{\frac{2}{T_S}}\cos\omega_c t + b_i\sqrt{\frac{2}{T_S}}\sin\omega_c t & \text{for } |t| \le T_S/2 \\ 0 & \text{elsewhere} \end{cases}$$

Choose $\sqrt{\mathrm{M}} = L$: integer , $\mathrm{L} = 2^{m/2}$ m/2 : integer $\rightarrow \mathrm{M} = \mathrm{L}^2 = 2^m$

$a_i = i_1 d - a_0, b_i = i_2 d - a_0$, where $i_1, i_2 = 1,2,\ldots, L$

$$a_0 = (1+L)\frac{d}{2}$$

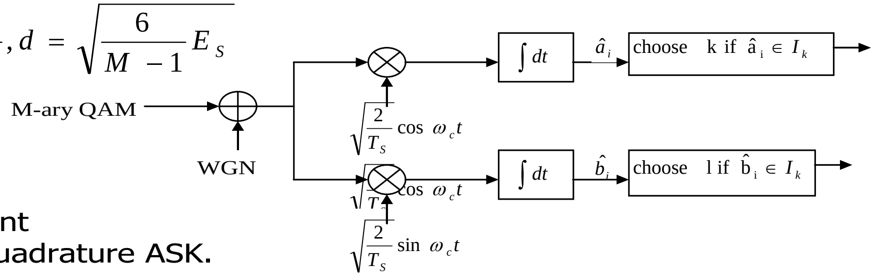
a_i and b_i are **independent**
a_i , b_i can take any
ONE of L possible values



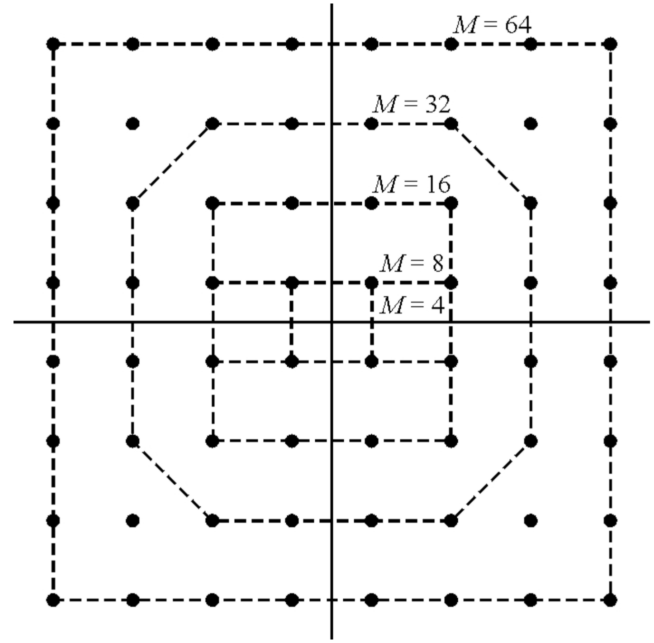
$$E_S = 2\left[\frac{2}{\sqrt{M}}\sum_{i=1}^{\sqrt{M}/2}\left[(2i-1)\frac{d}{2}\right]^2\right]$$

$$= \frac{d^2(M-1)}{6}, d = \sqrt{\frac{6}{M-1}E_S}$$

Receiver:



Treat it as
2 independent
inphase & quadrature ASK.



Each ASK has $P_{eASK} = (1 - \frac{1}{\sqrt{M}})\,erfc\left(\frac{d}{2\sqrt{N_0}}\right) \Rightarrow P_{eM-aryQAM} = 1 - [1 - P_{eASK}]^2$

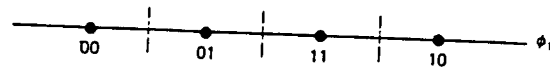
For large E_S/N_0, $P_{eASK} << 1$

$$P_{e,M-aryQAM} \approx 2P_{eASK} = 2\left(1 - \frac{1}{\sqrt{M}}\right)erfc\left(\frac{d}{2\sqrt{N_0}}\right) \approx 2\left(1 - \frac{1}{\sqrt{M}}\right)erfc\left(\sqrt{\frac{3}{2(M-1)}\frac{E_S}{N_0}}\right)$$

Notes: For M=4, L=2, 4QAM is 4PSK

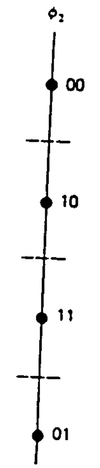
Probability of bit error (P_b) vs Probability of symbol error and Gray Coding

Examples of Gray coding:



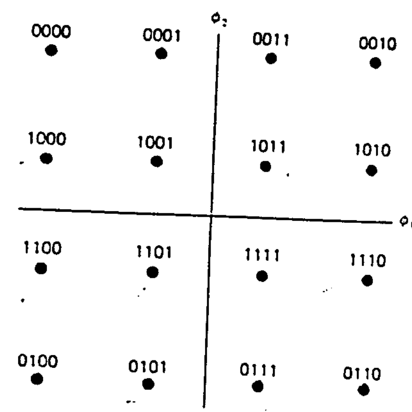
(a)

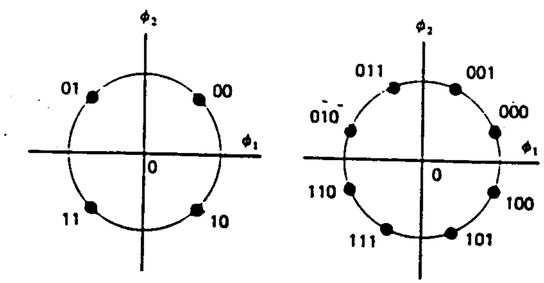
PAM



(b)

QAM





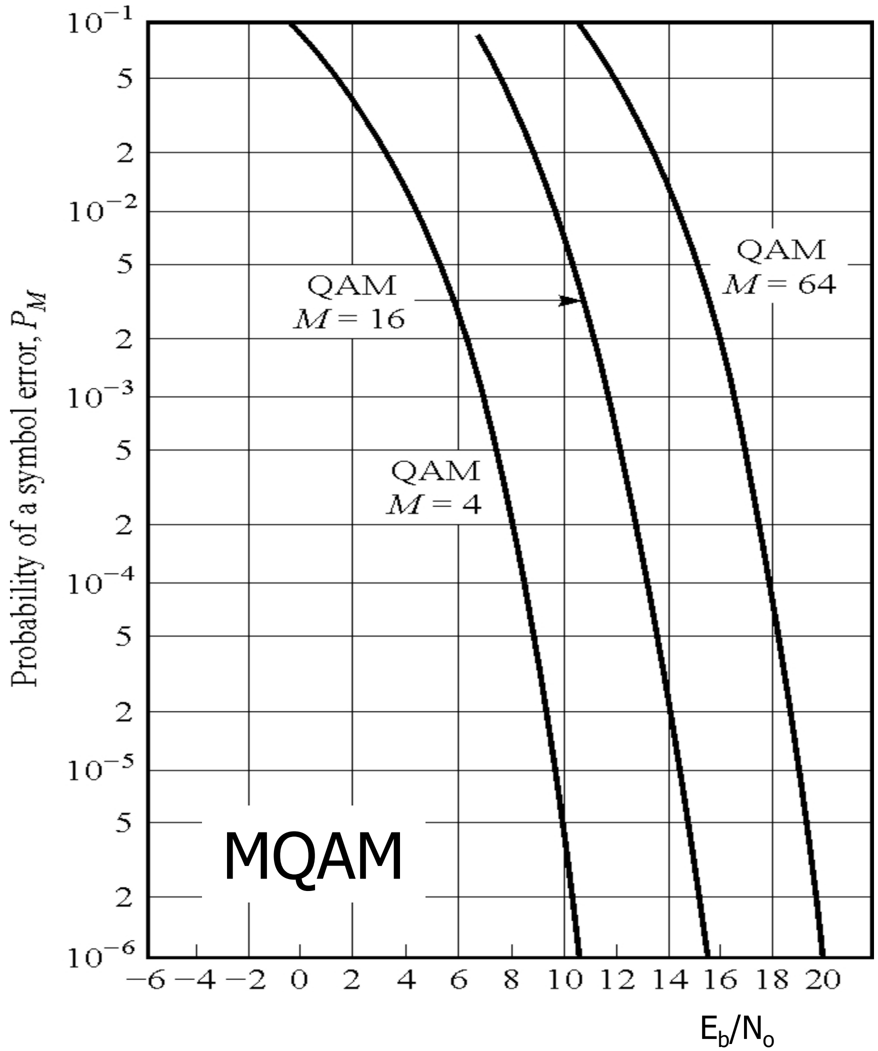
PSK

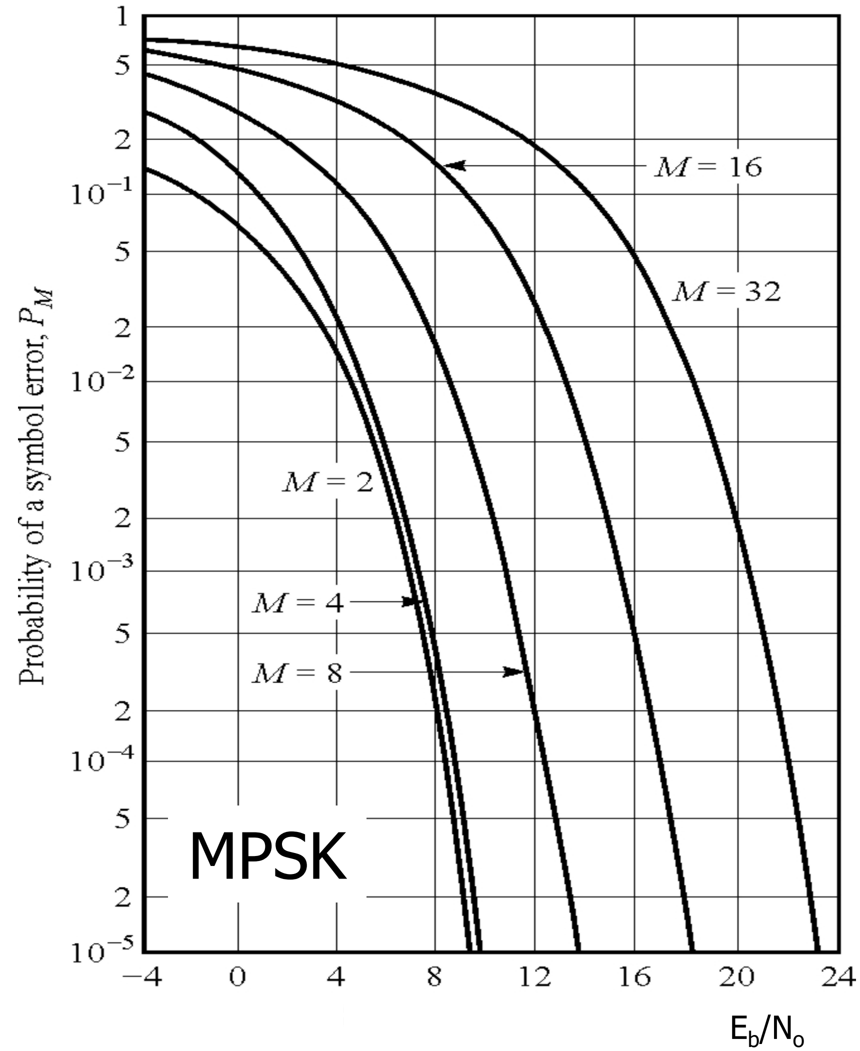
When a symbol error occurs, it is likely that the receiver takes the **adjacent** symbol (the symbol closest to the right one). Therefore, it is desirable to code the m-bit symbol in such a way that 2 adjacent symbols differ by **only one bit**. In this way, the average probability of bit error P_b is

$$P_b \approx \frac{1}{m} P_e = \frac{P_2}{\log_2 M}$$

PROBABILITY OF SYMBOL ERROR

M-QAM, M-PSK: BW-efficient but not power-efficient
For M>8, M-QAM outperforms M-PSK





Frequency Shift Keying (FSK)

M-ary orthogonal FSK signaling schemes are power-efficient but not bandwidth-efficient.

$$g_i(t) = \begin{cases} A\sqrt{\dfrac{2}{T_S}}\cos\omega_i t & 0 \le t \le T_S/2 \\ 0 & elsewhere \end{cases},$$

$$i = 1, 2, \ldots, M = 2^m, E_i = A^2 = \mathrm{E_S}$$

ORTHONORMAL M-ary FSK:

$$\int_0^{T_S} g_i(t).g_j(t)dt = 0 \text{ for i} \ne \text{j}$$

Note: Gray coding cannot be used for orthogonal FSK

$$\rightarrow (\omega_i - \omega_j)T_S = k\pi \text{ or } (f_i - f_j)T_S = k/2, \text{ k: integer}$$

$$\rightarrow d_{ij} = \left\| \underline{g}_i - \underline{g}_j \right\| = \sqrt{2A^2} = A\sqrt{2} = \sqrt{2\mathrm{E_S}} : \text{constant}$$

$$\rightarrow P_e \le \frac{1}{2}(M-1)erfc\sqrt{\frac{E_S}{2N_0}}$$

$$\rightarrow P_e \le \frac{1}{2}(M-1)erfc\sqrt{(\log_2 M)\frac{E_b}{2N_0}}$$